

Evaluation of a Polynomial

Let the polynomial $P(x)$ of degree n have the form

$$(20) \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Horner's method or **synthetic division** is a technique for evaluating polynomials. It can be thought of as nested multiplication. For example, a fifth-degree polynomial can be written in the nested multiplication form

$$P_5(x) = (((a_5x + a_4)x + a_3)x + a_2)x + a_1)x + a_0.$$

Theorem 1.13 (Horner's Method for Polynomial Evaluation). Assume that $P(x)$ is the polynomial given in equation (20) and $x = c$ is a number for which $P(c)$ is to be evaluated.

Set $b_n = a_n$ and compute

$$(21) \quad b_k = a_k + cb_{k+1} \quad \text{for } k = n - 1, n - 2, \dots, 1, 0;$$

then $b_0 = P(c)$. Moreover, if

$$(22) \quad Q_0(x) = b_nx^{n-1} + b_{n-1}x^{n-2} + \dots + b_3x^2 + b_2x + b_1,$$

then

$$(23) \quad P(x) = (x - c)Q_0(x) + R_0,$$

where $Q_0(x)$ is the quotient polynomial of degree $n - 1$ and $R_0 = b_0 = P(c)$ is the remainder.

Proof. Substituting the right side of equation (22) for $Q_0(x)$ and b_0 for R_0 in equation (23) yields

$$(24) \quad \begin{aligned} P(x) &= (x - c)(b_nx^{n-1} + b_{n-1}x^{n-2} + \dots + b_3x^2 + b_2x + b_1) + b_0 \\ &= b_nx^n + (b_{n-1} - cb_n)x^{n-1} + \dots + (b_2 - cb_3)x^2 \\ &\quad + (b_1 - cb_2)x + (b_0 - cb_1). \end{aligned}$$

The numbers b_k are determined by comparing the coefficients of x^k in equations (20) and (24), as shown in Table 1.1.

The value $P(c) = b_0$ is easily obtained by substituting $x = c$ into equation (22) and using the fact that $R_0 = b_0$:

$$(25) \quad P(c) = (c - c)Q_0(c) + R_0 = b_0. \quad \bullet$$

The recursive formula for b_k given in (21) is easy to implement with a computer. A simple algorithm is

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b(n) = a(n);
for k = n - 1: -1: 0
    b(k) = a(k) + c * b(k + 1);
end

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Table 1.1 Coefficients b_k for Horner's Method

x^k	Comparing (20) and (24)	Solving for b_k
x^n	$a_n = b_n$	$b_n = a_n$
x^{n-1}	$a_{n-1} = b_{n-1} - cb_n$	$b_{n-1} = a_{n-1} + cb_n$
\vdots	\vdots	\vdots
x^k	$a_k = b_k - cb_{k+1}$	$b_k = a_k + cb_{k+1}$
\vdots	\vdots	\vdots
x^0	$a_0 = b_0 - cb_1$	$b_0 = a_0 + cb_1$

Table 1.2 Horner's Table for the Synthetic Division Process

Input	a_n	a_{n-1}	a_{n-2}	\cdots	a_k	\cdots	a_2	a_1	a_0
c		xb_n	xb_{n-1}	\cdots	xb_{k+1}	\cdots	xb_3	xb_2	xb_1
	b_n	b_{n-1}	b_{n-2}	\cdots	b_k	\cdots	b_2	b_1	$b_0 = P(c)$
									Output

When Horner's method is performed by hand, it is easier to write the coefficients of $P(x)$ on a line and perform the calculation $b_k = a_k + cb_{k+1}$ below a_k in a column. The format for this procedure is illustrated in Table 1.2.

Example 1.9. Use synthetic division (Horner's method) to find $P(3)$ for the polynomial

$$P(x) = x^5 - 6x^4 + 8x^3 + 8x^2 + 4x - 40.$$

	a_5	a_4	a_3	a_2	a_1	a_0
Input	1	-6	8	8	4	-40
$c = 3$		3	-9	-3	15	57
	1	-3	-1	5	19	$17 = P(3) = b_0$
	b_5	b_4	b_3	b_2	b_1	Output

Therefore, $P(3) = 17$. ■

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John H. Mathews and Kurtis K. Fink

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JOHN H. MATHEWS • KURTIS D. FINK