

$n$ **Evaluation of a Polynomial**

Let the polynomial  $P(x)$  of degree  $n$  have the form

$$(20) \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

**Horner's method** or **synthetic division** is a technique for evaluating polynomials. It can be thought of as nested multiplication. For example, a fifth-degree polynomial can be written in the nested multiplication form

$$P_5(x) = (((a_5x + a_4)x + a_3)x + a_2)x + a_1)x + a_0.$$

**Theorem 1.13 (Horner's Method for Polynomial Evaluation).** Assume that  $P(x)$  is the polynomial given in equation (20) and  $x = c$  is a number for which  $P(c)$  is to be evaluated.

Set  $b_n = a_n$  and compute

$$(21) \quad b_k = a_k + cb_{k+1} \quad \text{for } k = n - 1, n - 2, \dots, 1, 0;$$

then  $b_0 = P(c)$ . Moreover, if

$$(22) \quad Q_0(x) = b_nx^{n-1} + b_{n-1}x^{n-2} + \dots + b_3x^2 + b_2x + b_1,$$

then

$$(23) \quad P(x) = (x - c)Q_0(x) + R_0,$$

where  $Q_0(x)$  is the quotient polynomial of degree  $n - 1$  and  $R_0 = b_0 = P(c)$  is the remainder.

*Proof.* Substituting the right side of equation (22) for  $Q_0(x)$  and  $b_0$  for  $R_0$  in equation (23) yields

$$(24) \quad \begin{aligned} P(x) &= (x - c)(b_nx^{n-1} + b_{n-1}x^{n-2} + \dots + b_3x^2 + b_2x + b_1) + b_0 \\ &= b_nx^n + (b_{n-1} - cb_n)x^{n-1} + \dots + (b_2 - cb_3)x^2 \\ &\quad + (b_1 - cb_2)x + (b_0 - cb_1). \end{aligned}$$

The numbers  $b_k$  are determined by comparing the coefficients of  $x^k$  in equations (20) and (24), as shown in Table 1.1.

The value  $P(c) = b_0$  is easily obtained by substituting  $x = c$  into equation (22) and using the fact that  $R_0 = b_0$ :

$$(25) \quad P(c) = (c - c)Q_0(c) + R_0 = b_0. \quad \bullet$$

The recursive formula for  $b_k$  given in (21) is easy to implement with a computer. A simple algorithm is

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b(n) = a(n);
for k = n - 1: -1: 0
    b(k) = a(k) + c * b(k + 1);
end

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**Table 1.1** Coefficients  $b_k$  for Horner's Method

$x^k$	Comparing (20) and (24)	Solving for $b_k$
$x^n$	$a_n = b_n$	$b_n = a_n$
$x^{n-1}$	$a_{n-1} = b_{n-1} - cb_n$	$b_{n-1} = a_{n-1} + cb_n$
$\vdots$	$\vdots$	$\vdots$
$x^k$	$a_k = b_k - cb_{k+1}$	$b_k = a_k + cb_{k+1}$
$\vdots$	$\vdots$	$\vdots$
$x^0$	$a_0 = b_0 - cb_1$	$b_0 = a_0 + cb_1$

**Table 1.2** Horner's Table for the Synthetic Division Process

Input	$a_n$	$a_{n-1}$	$a_{n-2}$	$\cdots$	$a_k$	$\cdots$	$a_2$	$a_1$	$a_0$
$c$		$xb_n$	$xb_{n-1}$	$\cdots$	$xb_{k+1}$	$\cdots$	$xb_3$	$xb_2$	$xb_1$
	$b_n$	$b_{n-1}$	$b_{n-2}$	$\cdots$	$b_k$	$\cdots$	$b_2$	$b_1$	$b_0 = P(c)$
									Output

When Horner's method is performed by hand, it is easier to write the coefficients of  $P(x)$  on a line and perform the calculation  $b_k = a_k + cb_{k+1}$  below  $a_k$  in a column. The format for this procedure is illustrated in Table 1.2.

**Example 1.9.** Use synthetic division (Horner's method) to find  $P(3)$  for the polynomial

$$P(x) = x^5 - 6x^4 + 8x^3 + 8x^2 + 4x - 40.$$

	$a_5$	$a_4$	$a_3$	$a_2$	$a_1$	$a_0$
Input	1	-6	8	8	4	-40
$c = 3$		3	-9	-3	15	57
	1	-3	-1	5	19	$17 = P(3) = b_0$
	$b_5$	$b_4$	$b_3$	$b_2$	$b_1$	Output

Therefore,  $P(3) = 17$ . ■

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