The Fibonacci Search

Fibonacci Search

In the golden ratio search two function evaluations are made at the first iteration and then only one function evaluation is made for each subsequent iteration. The value of

r remains constant on each subinterval and the search is terminated at the kth subinterval, provided that $|b_k - a_k|$ or $|f(b_k) - f(a_k)|$ satisfies predefined tolerances. The ***Fibonacci search method*** differs from the golden ratio method in that the value of r is not constant on each subinterval. Additionally, the number of subintervals (iterations) is predetermined and based on the specified tolerances.

The Fibonacci search is based on the sequence of Fibonacci numbers $\{F_k\}_{k=0}^{\infty}$ defined by the equations

$$F_0 = 0, \; F_1 = 1 \tag{5}$$

$$F_n = F_{n-1} + F_{n-2} \tag{6}$$

for $n = 2, 3, \ldots$. Thus the Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

Assume we are given a function $f(x)$ that is unimodal on the interval $[a_0, b_0]$. As in the golden ratio search a value r_0 $(1/2 < r_0 < 1)$ is selected so that both of the interior points c_0 and d_0 will be used in the next subinterval and there will be only one new function evaluation. Without loss of generality assume that $f(c_0) > f(d_0)$. It follows that $a_1 = a_0$, $b_1 = d_0$, and $d_1 = c_0$ (see Figure 8.4). If there is to be only one new function evaluation, then we select r_1 $(1/2 < r_1 < 1)$ for the subinterval $[a_1, b_1]$, such that

$$\begin{aligned}
d_0 - c_0 &= b_1 - d_1 \\
(2r_0 - 1)(b_0 - a_0) &= (1 - r_1)(b_1 - a_1) \\
(2r_0 - 1)(b_0 - a_0) &= (1 - r_1)(r_0(b_0 - a_0)) \\
2r_0 - 1 &= (1 - r_1)r_0 \\
r_1 &= \frac{1 - r_0}{r_0}.
\end{aligned}$$

Substituting $r_0 = F_{n-1}/F_n$, $n \geq 4$, into this last equation yields

$$\begin{aligned}
r_1 &= \frac{1 - \dfrac{F_{n-1}}{F_n}}{\dfrac{F_{n-1}}{F_n}} \\
&= \frac{F_n - F_{n-1}}{F_{n-1}} \\
&= \frac{F_{n-2}}{F_{n-1}}
\end{aligned}$$

since, by equation (6), $F_n = F_{n-1} + F_{n-2}$.

Reasoning inductively, it follows that the Fibonacci search can be begun with $r_0 = F_{n-1}/F_n$ and continued using $r_k = F_{n-1-k}/F_{n-k}$ for $k = 1, 2, \ldots, n-3$. Note that $r_{n-3} = F_2/F_3 = 1/2$, thus no new points can be added at this stage. Therefore, there are a total of $(n-3) + 1 = n - 2$ steps in this process.

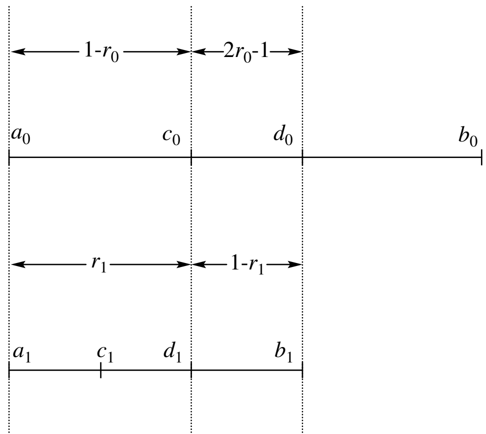

Figure 8.4 The Fibonacci search intervals $[a_0, b_0]$ and $[a_1, b_1]$.

The $(k+1)$st subinterval is obtained by reducing the length of the kth subinterval by a factor of $r_k = F_{n-1-k}/F_{n-k}$. The length of the last subinterval is

$$\frac{F_{n-1}F_{n-2}\cdots F_2}{F_nF_{n-1}\cdots F_3}(b_0 - a_0) = \frac{F_2}{F_n}(b_0 - a_0)$$
$$= \frac{1}{F_n}(b_0 - a_0) = \frac{b_0 - a_0}{F_n}.$$

If the absissa of the minimum is to be found with a tolerance of ϵ, then we need to find the smallest value of n such that

$$\frac{b_0 - a_0}{F_n} < \epsilon \quad \text{or} \quad F_n > \frac{b_0 - a_0}{\epsilon}. \tag{7}$$

The interior points c_k and d_k of the kth subinterval $[a_k, b_k]$ are found, as needed, using the formulas

$$c_k = a_k + \left(1 - \frac{F_{n-k-1}}{F_{n-k}}\right)(b_k - a_k) \tag{8}$$

$$d_k = a_k + \frac{F_{n-k-1}}{F_{n-k}}(b_k - a_k). \tag{9}$$

Note. the value of n used in formulas (8) and (9) is found using inequality (7).

Each iteration requires the determination of two new interior points, one from the previous iteration and the second from formula (8) or (9). When $r_0 = F_2/F_3 = 1/2$, the two interior points will be concurrent in the middle of the interval. To distinguish the two interior points a small distinguishability constant, e, is introducted. Thus when formula (8) or (9) is used, the coefficients of $(b_k - a_k)$ are $1/2 - e$ or $1/2 + e$, respectively.

Example 8.3. Find the minimum of the function $f(x) = x^2 - \sin(x)$ on the interval [0, 1] using the Fibonacci search method. Use a tolerance of $\epsilon = 10^{-4}$ and the distinguishability constant $e = 0.01$.

The smallest Fibonacci number satisfying

$$F_n > \frac{b_0 - a_0}{\epsilon} = \frac{1 - 0}{10^{-4}} = 10{,}000,$$

is $F_{21} = 10{,}946$. Thus $n = 21$. Let $a_0 = 0$ and $b_0 = 1$. Formulas (8) and (9) yield

$$c_0 = 0 + \left(1 - \frac{F_{20}}{F_{21}}\right)(1 - 0) \approx 0.3819660$$
$$d_0 = 0 + \frac{F_{20}}{F_{21}}(1 - 0) \approx 0.6180340.$$

We set $a_1 = a_0$, $b_1 = d_0$, and $d_1 = c_0$, since $f(0.3819660) = -0.2268475$ and $f(0.6180340) = -0.1974679$ ($f(d_0) \geq f(c_0)$). The new subinterval containing the abscissa of the minimum of f is $[a_1, b_1] = [0, 0.6180340]$. Now use formula (8) to calculate the interior point c_1:

$$\begin{aligned} c_1 &= a_1 + \left(1 - \frac{F_{21-1-1}}{F_{21-1}}\right)(b_1 - a_1) \\ &= 0 + \left(1 - \frac{F_{19}}{F_{20}}\right)(0.6180340 - 0) \\ &\approx 0.2360680. \end{aligned}$$

Now compute and compare $f(c_1)$ and $f(d_1)$ to determine the new subinterval $[a_2, b_2]$, and continue the iteration process. Some of the computations are shown in Table 8.3.

At the seventeenth iteration the interval has been narrowed down to $[a_{17}, b_{17}] = [0.4501188, 0.4503928]$, where $c_{17} = 0.4502101$, $d_{17} = 0.4503105$, and $f(d_{17}) \geq f(c_{17})$. Thus $[a_{18}, b_{18}] = [0.4501188, 0.4503015]$ and $d_{18} = 0.4502101$. At this stage the multiplier is $r_{18} = 1 - F_2/F_3 = 1 - 1/2 = 1/2$ and the distinguishability constant $e = 0.01$ is used to calculate c_{18}:

$$\begin{aligned} c_{18} &= a_{18} + (0.5 - 0.01)(b_{18} - a_{18}) \\ &= 0.4501188 - 0.49(0.450315 - 0.4501188) \\ &\approx 0.4502083. \end{aligned}$$

Since $f(d_{18}) \geq f(c_{18})$, the final subinterval is $[a_{19}, b_{19}] = [0.4501188, 0.4502101]$. This interval has width 0.0000913. We choose to report the abscissa of the minimum as the midpoint of this interval. Therefore, the minimum value is $f(0.4501645) = -0.2324656$. ■

Both the Fibonacci and golden ratio search methods can be applied in cases where $f(x)$ is not differentiable. It should be noted that when n is small the Fibonacci method is more efficient than the golden ratio method. However, for n large the two methods are almost identical.

Table 8.3 Fibonacci Search for the Minimum of $f(x) = x^2 - \sin(x)$

k	a_k	c_k	d_k	b_k
0	0.0000000	0.3819660	0.6180340	1.0000000
1	0.0000000	0.2360680	0.3819660	0.6180340
2	0.2360680	0.3819660	0.4721359	0.6180340
3	0.3819660	0.4721359	0.5278641	0.6180340
4	0.3819660	0.4376941	0.4721359	0.5278641
⋮	⋮	⋮	⋮	⋮
16	0.4499360	0.4501188	0.4502102	0.4503928
17	0.4501188	0.4502101	0.4503015	0.4503928
18	0.4501188	0.4502083	0.4502101	0.4503015

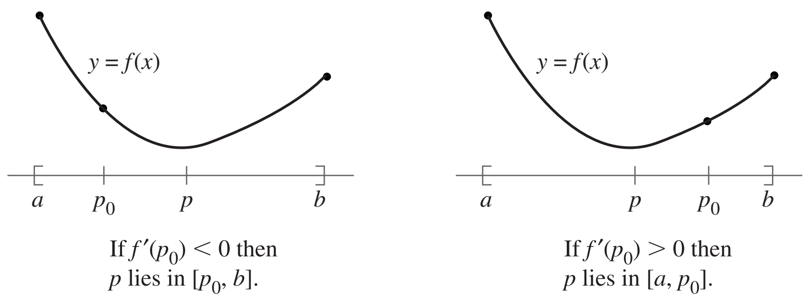

Figure 8.5 Using $f'(x)$ to find the minimum value of the unimodal function $f(x)$ on the interval $[a, b]$.

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