

# 1

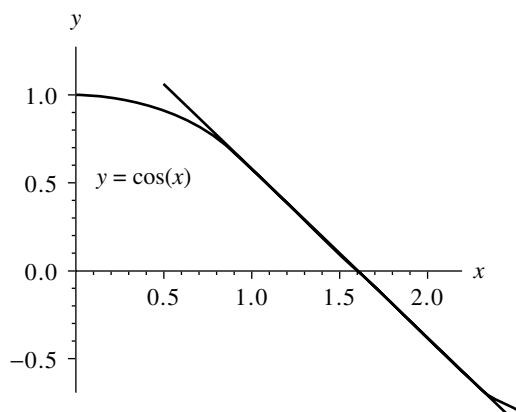
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## Preliminaries

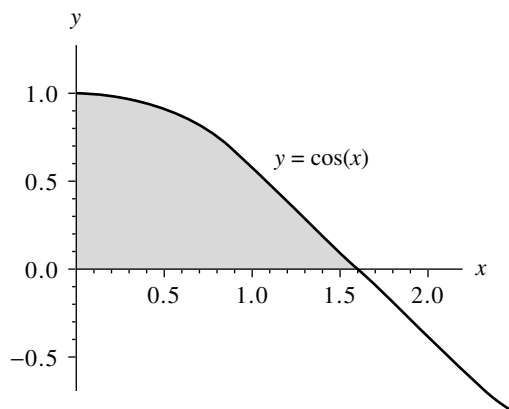
Consider the function  $f(x) = \cos(x)$ , its derivative  $f'(x) = -\sin(x)$ , and its antiderivative  $F(x) = \sin(x) + C$ . These formulas were studied in calculus. The former is used to determine the slope  $m = f'(x_0)$  of the curve  $y = f(x)$  at a point  $(x_0, f(x_0))$ , and the latter is used to compute the area under the curve for  $a \leq x \leq b$ .

The slope at the point  $(\pi/2, 0)$  is  $m = f'(\pi/2) = -1$  and can be used to find the tangent line at this point (see Figure 1.1(a)):

$$y_{\text{tan}} = m \left( x - \frac{\pi}{2} \right) + 0 = f' \left( \frac{\pi}{2} \right) \left( x - \frac{\pi}{2} \right) = -x + \frac{\pi}{2}.$$



**Figure 1.1** (a) The tangent line to the curve  $y = \cos(x)$  at the point  $(\pi/2, 0)$ .



**Figure 1.1** (b) The area under the curve  $y = \cos(x)$  over the interval  $[0, \pi/2]$ .

The area under the curve for  $0 \leq x \leq \pi/2$  is computed using an integral (see Figure 1.1(b)):

$$\text{area} = \int_0^{\pi/2} \cos(x) dx = F\left(\frac{\pi}{2}\right) - F(0) = \sin\left(\frac{\pi}{2}\right) - 0 = 1.$$

These are some of the results that we will need to use from calculus.

## 1.1 Review of Calculus

It is assumed that the reader is familiar with the notation and subject matter covered in the undergraduate calculus sequence. This should have included the topics of limits, continuity, differentiation, integration, sequences, and series. Throughout the book we refer to the following results.

### Limits and Continuity

**Definition 1.1.** Assume that  $f(x)$  is defined on an open interval containing  $x = x_0$ , except possibly at  $x = x_0$  itself. Then  $f$  is said to have the *limit*  $L$  at  $x = x_0$ , and we write

$$(1) \quad \lim_{x \rightarrow x_0} f(x) = L,$$

if given any  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $|f(x) - L| < \epsilon$  whenever  $0 < |x - x_0| < \delta$ . When the  $h$ -increment notation  $x = x_0 + h$  is used, equation (1) becomes

$$(2) \quad \lim_{h \rightarrow 0} f(x_0 + h) = L. \quad \blacktriangle$$

**Definition 1.2.** Assume that  $f(x)$  is defined on an open interval containing  $x = x_0$ . Then  $f$  is said to be **continuous at  $x = x_0$**  if

$$(3) \quad \lim_{x \rightarrow x_0} f(x) = f(x_0).$$

The function  $f$  is said to be continuous on a set  $S$  if it is continuous at each point  $x \in S$ . The notation  $C^n(S)$  stands for the set of all functions  $f$  such that  $f$  and its first  $n$  derivatives are continuous on  $S$ . When  $S$  is an interval, say  $[a, b]$ , then the notation  $C^n[a, b]$  is used. As an example, consider the function  $f(x) = x^{4/3}$  on the interval  $[-1, 1]$ . Clearly,  $f(x)$  and  $f'(x) = (4/3)x^{1/3}$  are continuous on  $[-1, 1]$ , while  $f''(x) = (4/9)x^{-2/3}$  is not continuous at  $x = 0$ . ▲

**Definition 1.3.** Suppose that  $\{x_n\}_{n=1}^{\infty}$  is an infinite sequence. Then the sequence is said to have the **limit  $L$** , and we write

$$(4) \quad \lim_{n \rightarrow \infty} x_n = L,$$

if given any  $\epsilon > 0$ , there exists a positive integer  $N = N(\epsilon)$  such that  $n > N$  implies that  $|x_n - L| < \epsilon$ . ▲

When a sequence has a limit, we say that it is a **convergent sequence**. Another commonly used notation is “ $x_n \rightarrow L$  as  $n \rightarrow \infty$ .” Equation (4) is equivalent to

$$(5) \quad \lim_{n \rightarrow \infty} (x_n - L) = 0.$$

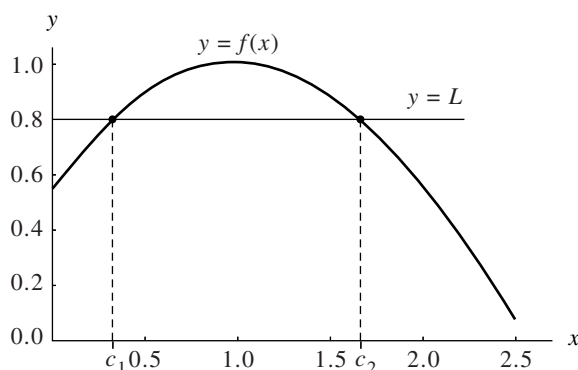
Thus we can view the sequence  $\{\epsilon_n\}_{n=1}^{\infty} = \{x_n - L\}_{n=1}^{\infty}$  as an **error sequence**. The following theorem relates the concepts of continuity and convergent sequence.

**Theorem 1.1.** Assume that  $f(x)$  is defined on the set  $S$  and  $x_0 \in S$ . The following statements are equivalent:

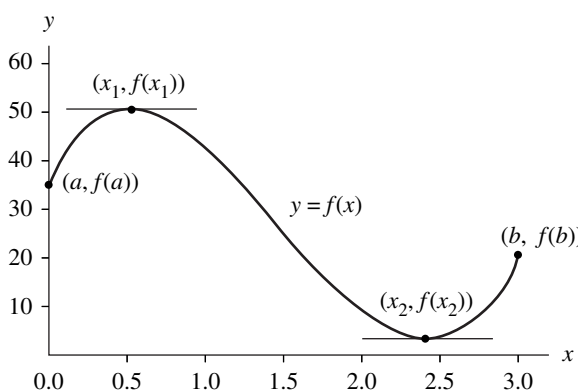
- (6) (a) The function  $f$  is continuous at  $x_0$ .  
 (b) If  $\lim_{n \rightarrow \infty} x_n = x_0$ , then  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .

**Theorem 1.2 (Intermediate Value Theorem).** Assume that  $f \in C[a, b]$  and  $L$  is any number between  $f(a)$  and  $f(b)$ . Then there exists a number  $c$ , with  $c \in (a, b)$ , such that  $f(c) = L$ .

**Example 1.1.** The function  $f(x) = \cos(x - 1)$  is continuous over  $[0, 1]$ , and the constant  $L = 0.8 \in (\cos(0), \cos(1))$ . The solution to  $f(x) = 0.8$  over  $[0, 1]$  is  $c_1 = 0.356499$ . Similarly,  $f(x)$  is continuous over  $[1, 2.5]$ , and  $L = 0.8 \in (\cos(2.5), \cos(1))$ . The solution to  $f(x) = 0.8$  over  $[1, 2.5]$  is  $c_2 = 1.643502$ . These two cases are shown in Figure 1.2. ■



**Figure 1.2** The intermediate value theorem applied to the function  $f(x) = \cos(x - 1)$  over  $[0, 1]$  and over the interval  $[1, 2.5]$ .



**Figure 1.3** The extreme value theorem applied to the function  $f(x) = 35 + 59.5x - 66.5x^2 + 15x^3$  over the interval  $[0, 3]$ .

**Theorem 1.3 (Extreme Value Theorem for a Continuous Function).** Assume that  $f \in C[a, b]$ . Then there exists a lower bound  $M_1$ , an upper bound  $M_2$ , and two numbers  $x_1, x_2 \in [a, b]$  such that

$$(7) \quad M_1 = f(x_1) \leq f(x) \leq f(x_2) = M_2 \quad \text{whenever } x \in [a, b].$$

We sometimes express this by writing

$$(8) \quad M_1 = f(x_1) = \min_{a \leq x \leq b} \{f(x)\} \quad \text{and} \quad M_2 = f(x_2) = \max_{a \leq x \leq b} \{f(x)\}.$$

## Differentiable Functions

**Definition 1.4.** Assume that  $f(x)$  is defined on an open interval containing  $x_0$ . Then  $f$  is said to be *differentiable* at  $x_0$  if

$$(9) \quad \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. When this limit exists, it is denoted by  $f'(x_0)$  and is called the *derivative* of  $f$  at  $x_0$ . An equivalent way to express this limit is to use the  $h$ -increment notation:

$$(10) \quad \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

A function that has a derivative at each point in a set  $S$  is said to be *differentiable* on  $S$ . Note that the number  $m = f'(x_0)$  is the slope of the tangent line to the graph of the function  $y = f(x)$  at the point  $(x_0, f(x_0))$ . ▲

**Theorem 1.4.** If  $f(x)$  is differentiable at  $x = x_0$ , then  $f(x)$  is continuous at  $x = x_0$ .

It follows from Theorem 1.3 that if a function  $f$  is differentiable on a closed interval  $[a, b]$ , then its extreme values occur at the endpoints of the interval or at the critical points (solutions of  $f'(x) = 0$ ) in the open interval  $(a, b)$ .

**Example 1.2.** The function  $f(x) = 15x^3 - 66.5x^2 + 59.5x + 35$  is differentiable on  $[0, 3]$ . The solutions to  $f'(x) = 45x^2 - 123x + 59.5 = 0$  are  $x_1 = 0.54955$  and  $x_2 = 2.40601$ . The maximum and minimum values of  $f$  on  $[0, 3]$  are:

$$\min\{f(0), f(3), f(x_1), f(x_2)\} = \min\{35, 20, 50.10438, 2.11850\} = 2.11850$$

and

$$\max\{f(0), f(3), f(x_1), f(x_2)\} = \max\{35, 20, 50.10438, 2.11850\} = 50.10438$$

(see Figure 1.3). ■

**Theorem 1.5 (Rolle's Theorem).** Assume that  $f \in C[a, b]$  and that  $f'(x)$  exists for all  $x \in (a, b)$ . If  $f(a) = f(b) = 0$ , then there exists a number  $c$ , with  $c \in (a, b)$ , such that  $f'(c) = 0$ .

**Theorem 1.6 (Mean Value Theorem).** Assume that  $f \in C[a, b]$  and that  $f'(x)$  exists for all  $x \in (a, b)$ . Then there exists a number  $c$ , with  $c \in (a, b)$ , such that

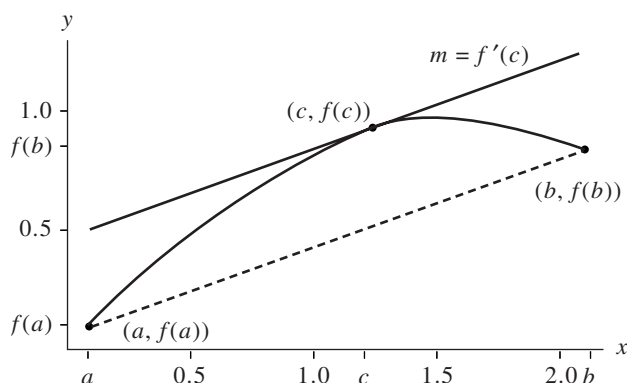
$$(11) \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, the mean value theorem says that there is at least one number  $c \in (a, b)$  such that the slope of the tangent line to the graph of  $y = f(x)$  at the point  $(c, f(c))$  equals the slope of the secant line through the points  $(a, f(a))$  and  $(b, f(b))$ .

**Example 1.3.** The function  $f(x) = \sin(x)$  is continuous on the closed interval  $[0.1, 2.1]$  and differentiable on the open interval  $(0.1, 2.1)$ . Thus, by the mean value theorem, there is a number  $c$  such that

$$f'(c) = \frac{f(2.1) - f(0.1)}{2.1 - 0.1} = \frac{0.863209 - 0.099833}{2.1 - 0.1} = 0.381688.$$

The solution to  $f'(c) = \cos(c) = 0.381688$  in the interval  $(0.1, 2.1)$  is  $c = 1.179174$ . The graphs of  $f(x)$ , the secant line  $y = 0.381688x + 0.099833$ , and the tangent line  $y = 0.381688x + 0.474215$  are shown in Figure 1.4. ■



**Figure 1.4** The mean value theorem applied to  $f(x) = \sin(x)$  over the interval  $[0.1, 2.1]$ .

**Theorem 1.7 (Generalized Rolle's Theorem).** Assume that  $f \in C[a, b]$  and that  $f'(x), f''(x), \dots, f^{(n)}(x)$  exist over  $(a, b)$  and  $x_0, x_1, \dots, x_n \in [a, b]$ . If  $f(x_j) = 0$  for  $j = 0, 1, \dots, n$ , then there exists a number  $c$ , with  $c \in (a, b)$ , such that  $f^{(n)}(c) = 0$ .

## Integrals

**Theorem 1.8 (First Fundamental Theorem).** If  $f$  is continuous over  $[a, b]$  and  $F$  is any antiderivative of  $f$  on  $[a, b]$ , then

$$(12) \quad \int_a^b f(x) dx = F(b) - F(a) \quad \text{where } F'(x) = f(x).$$

**Theorem 1.9 (Second Fundamental Theorem).** If  $f$  is continuous over  $[a, b]$  and  $x \in (a, b)$ , then

$$(13) \quad \frac{d}{dx} \int_a^x f(t) dt = f(x).$$

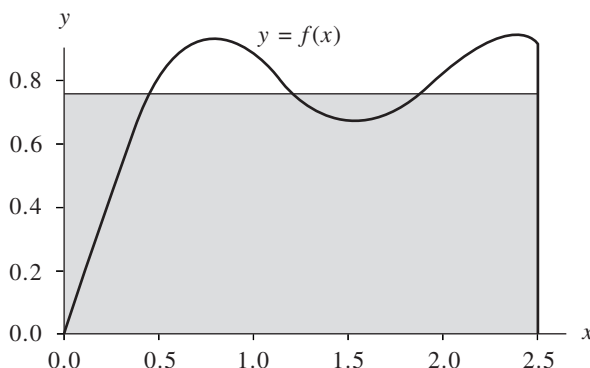
**Example 1.4.** The function  $f(x) = \cos(x)$  satisfies the hypotheses of Theorem 1.9 over the interval  $[0, \pi/2]$ ; thus by the chain rule

$$\frac{d}{dx} \int_0^{x^2} \cos(t) dt = \cos(x^2)(x^2)' = 2x \cos(x^2). \quad \blacksquare$$

**Theorem 1.10 (Mean Value Theorem for Integrals).** Assume that  $f \in C[a, b]$ . Then there exists a number  $c$ , with  $c \in (a, b)$ , such that

$$\frac{1}{b-a} \int_a^b f(x) dx = f(c).$$

The value  $f(c)$  is the average value of  $f$  over the interval  $[a, b]$ .



**Figure 1.5** The mean value theorem for integrals applied to  $f(x) = \sin(x) + \frac{1}{3} \sin(3x)$  over the interval  $[0, 2.5]$ .

**Example 1.5.** The function  $f(x) = \sin(x) + \frac{1}{3} \sin(3x)$  satisfies the hypotheses of Theorem 1.10 over the interval  $[0, 2.5]$ . An antiderivative of  $f(x)$  is  $F(x) = -\cos(x) - \frac{1}{9} \cos(3x)$ . The average value of the function  $f(x)$  over the interval  $[0, 2.5]$  is

$$\begin{aligned} \frac{1}{2.5 - 0} \int_0^{2.5} f(x) dx &= \frac{F(2.5) - F(0)}{2.5} = \frac{0.762629 - (-1.111111)}{2.5} \\ &= \frac{1.873740}{2.5} = 0.749496. \end{aligned}$$

There are three solutions to the equation  $f(c) = 0.749496$  over the interval  $[0, 2.5]$ :  $c_1 = 0.440566$ ,  $c_2 = 1.268010$ , and  $c_3 = 1.873583$ . The area of the rectangle with base  $b - a = 2.5$  and height  $f(c_j) = 0.749496$  is  $f(c_j)(b - a) = 1.873740$ . The area of the rectangle has the same numerical value as the integral of  $f(x)$  taken over the interval  $[0, 2.5]$ . A comparison of the area under the curve  $y = f(x)$  and that of the rectangle can be seen in Figure 1.5. ■

**Theorem 1.11 (Weighted Integral Mean Value Theorem).** Assume that  $f, g \in C[a, b]$  and  $g(x) \geq 0$  for  $x \in [a, b]$ . Then there exists a number  $c$ , with  $c \in (a, b)$ , such that

$$(14) \quad \int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx.$$

**Example 1.6.** The functions  $f(x) = \sin(x)$  and  $g(x) = x^2$  satisfy the hypotheses of Theorem 1.11 over the interval  $[0, \pi/2]$ . Thus there exists a number  $c$  such that

$$\sin(c) = \frac{\int_0^{\pi/2} x^2 \sin(x) dx}{\int_0^{\pi/2} x^2 dx} = \frac{1.14159}{1.29193} = 0.883631$$

or  $c = \sin^{-1}(0.883631) = 1.08356$ . ■

## Series

**Definition 1.5.** Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence. Then  $\sum_{n=1}^{\infty} a_n$  is an infinite series. The  $n$ th partial sum is  $S_n = \sum_{k=1}^n a_k$ . The infinite series **converges** if and only if the sequence  $\{S_n\}_{n=1}^{\infty}$  converges to a limit  $S$ , that is,

$$(15) \quad \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = S.$$

If a series does not converge, we say that it **diverges**. ▲

**Example 1.7.** Consider the infinite sequence  $\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{n(n+1)} \right\}_{n=1}^{\infty}$ . Then the  $n$ th partial sum is

$$S_n = \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}.$$

Therefore, the **sum** of the infinite series is

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) = 1. \quad \blacksquare$$

**Theorem 1.12 (Taylor's Theorem).** Assume that  $f \in C^{n+1}[a, b]$  and let  $x_0 \in [a, b]$ . Then, for every  $x \in (a, b)$ , there exists a number  $c = c(x)$  (the value of  $c$  depends on the value of  $x$ ) that lies between  $x_0$  and  $x$  such that

$$(16) \quad f(x) = P_n(x) + R_n(x),$$

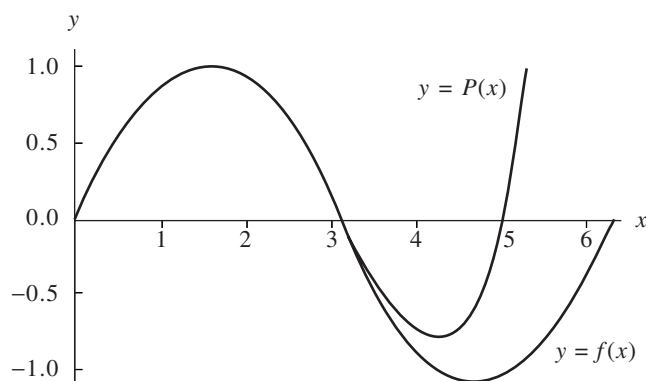
where

$$(17) \quad P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

and

$$(18) \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

**Example 1.8.** The function  $f(x) = \sin(x)$  satisfies the hypotheses of Theorem 1.12. The Taylor polynomial  $P_n(x)$  of degree  $n = 9$  expanded about  $x_0 = 0$  is obtained by evaluating



**Figure 1.6** The graph of  $f(x) = \sin(x)$  and the Taylor polynomial  $P_9(x) = x - x^3/3! + x^5/5! - x^7/7! + x^9/9!$ .

the following derivatives at  $x = 0$  and substituting the numerical values into formula (17).

$$\begin{aligned} f(x) &= \sin(x), & f(0) &= 0, \\ f'(x) &= \cos(x), & f'(0) &= 1, \\ f''(x) &= -\sin(x), & f''(0) &= 0, \\ f^{(3)}(x) &= -\cos(x), & f^{(3)}(0) &= -1, \\ &\vdots & &\vdots \\ f^{(9)}(x) &= \cos(x), & f^{(9)}(0) &= 1, \end{aligned}$$

$$P_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}.$$

A graph of both  $f$  and  $P_9$  over the interval  $[0, 2\pi]$  is shown in Figure 1.6. ■

**Corollary 1.1.** If  $P_n(x)$  is the Taylor polynomial of degree  $n$  given in Theorem 1.12, then

$$(19) \quad P_n^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 0, 1, \dots, n.$$

### Evaluation of a Polynomial

Let the polynomial  $P(x)$  of degree  $n$  have the form

$$(20) \quad P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0.$$

**Horner's method** or **synthetic division** is a technique for evaluating polynomials. It can be thought of as nested multiplication. For example, a fifth-degree polynomial can be written in the nested multiplication form

$$P_5(x) = (((a_5x + a_4)x + a_3)x + a_2)x + a_1)x + a_0.$$

**Theorem 1.13 (Horner's Method for Polynomial Evaluation).** Assume that  $P(x)$  is the polynomial given in equation (20) and  $x = c$  is a number for which  $P(c)$  is to be evaluated.

Set  $b_n = a_n$  and compute

$$(21) \quad b_k = a_k + cb_{k+1} \quad \text{for } k = n - 1, n - 2, \dots, 1, 0;$$

then  $b_0 = P(c)$ . Moreover, if

$$(22) \quad Q_0(x) = b_nx^{n-1} + b_{n-1}x^{n-2} + \dots + b_3x^2 + b_2x + b_1,$$

then

$$(23) \quad P(x) = (x - c)Q_0(x) + R_0,$$

where  $Q_0(x)$  is the quotient polynomial of degree  $n - 1$  and  $R_0 = b_0 = P(c)$  is the remainder.

*Proof.* Substituting the right side of equation (22) for  $Q_0(x)$  and  $b_0$  for  $R_0$  in equation (23) yields

$$(24) \quad \begin{aligned} P(x) &= (x - c)(b_nx^{n-1} + b_{n-1}x^{n-2} + \dots + b_3x^2 + b_2x + b_1) + b_0 \\ &= b_nx^n + (b_{n-1} - cb_n)x^{n-1} + \dots + (b_2 - cb_3)x^2 \\ &\quad + (b_1 - cb_2)x + (b_0 - cb_1). \end{aligned}$$

The numbers  $b_k$  are determined by comparing the coefficients of  $x^k$  in equations (20) and (24), as shown in Table 1.1.

The value  $P(c) = b_0$  is easily obtained by substituting  $x = c$  into equation (22) and using the fact that  $R_0 = b_0$ :

$$(25) \quad P(c) = (c - c)Q_0(c) + R_0 = b_0. \quad \bullet$$

The recursive formula for  $b_k$  given in (21) is easy to implement with a computer. A simple algorithm is

```

b(n) = a(n);
for k = n - 1: -1: 0
    b(k) = a(k) + c * b(k + 1);
end

```

**Table 1.1** Coefficients  $b_k$  for Horner's Method

$x^k$	Comparing (20) and (24)	Solving for $b_k$
$x^n$	$a_n = b_n$	$b_n = a_n$
$x^{n-1}$	$a_{n-1} = b_{n-1} - cb_n$	$b_{n-1} = a_{n-1} + cb_n$
$\vdots$	$\vdots$	$\vdots$
$x^k$	$a_k = b_k - cb_{k+1}$	$b_k = a_k + cb_{k+1}$
$\vdots$	$\vdots$	$\vdots$
$x^0$	$a_0 = b_0 - cb_1$	$b_0 = a_0 + cb_1$

**Table 1.2** Horner's Table for the Synthetic Division Process

Input	$a_n$	$a_{n-1}$	$a_{n-2}$	$\cdots$	$a_k$	$\cdots$	$a_2$	$a_1$	$a_0$
$c$		$xb_n$	$xb_{n-1}$	$\cdots$	$xb_{k+1}$	$\cdots$	$xb_3$	$xb_2$	$xb_1$
	$b_n$	$b_{n-1}$	$b_{n-2}$	$\cdots$	$b_k$	$\cdots$	$b_2$	$b_1$	$b_0 = P(c)$
									Output

When Horner's method is performed by hand, it is easier to write the coefficients of  $P(x)$  on a line and perform the calculation  $b_k = a_k + cb_{k+1}$  below  $a_k$  in a column. The format for this procedure is illustrated in Table 1.2.

**Example 1.9.** Use synthetic division (Horner's method) to find  $P(3)$  for the polynomial

$$P(x) = x^5 - 6x^4 + 8x^3 + 8x^2 + 4x - 40.$$

	$a_5$	$a_4$	$a_3$	$a_2$	$a_1$	$a_0$
Input	1	-6	8	8	4	-40
$c = 3$		3	-9	-3	15	57
	1	-3	-1	5	19	$17 = P(3) = b_0$
	$b_5$	$b_4$	$b_3$	$b_2$	$b_1$	Output

Therefore,  $P(3) = 17$ . ■

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